

Feedback under Control-Dependent Markov Disturbances: A Discrete Maximum Principle

Pham T. Nhu

*Institute of Computer Science and Cybernetics, P.O. Box 634, Bo Ho, Hanoi,
Vietnam 10 000*

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A necessary condition of a stochastic maximum principle type is given for an optimal solution of discrete-time feedback control problems under Markov disturbances depending on control parameters. © 1996 Academic Press, Inc.

1. INTRODUCTION

For a discrete-time system which operates under several modes with the change described by a finite state Markov disturbance, the dependence of the disturbance on control parameters in the sense that

$$E[l(\eta_{t+1}) \mid \eta_t = i] - \sum_{j=1}^S l(j)p(i, j, u_t) = 0,$$

$$l: I = \{1, 2, \dots, S\} \mapsto [0, +\infty),$$

arises in the situation when the change between the modes is influenced by acting of control strategies. From a mathematical programming viewpoint, the complication arising here is to define an optimal control in the presence of these equality constraints defined by all functions $l: I \mapsto R^+$. On the other hand, the disturbance with such a dependence will be able to change jumping probabilities $p(i, j, u_t)$ and in this case, finding an optimal strategy becomes more complicated because the disturbance is able to “recognize” the control strategies and changes the probabilities of jumping to more or less expensive modes to prevent attainment of gains accordingly.

For such systems, Bellman’s dynamic programming principle was given in [6] and Pontriagin’s maximum principle was discussed in [7] for the class

\mathcal{U} of nonanticipating control strategies. Their applications to controlled Markov chains (see, e.g., [3]) were also given in [6, 7]. Due to the stochastic character of the problem, one of the main difficulties is to define an optimal control in feedback form. In the present paper, the optimization problem is considered only in the class \mathcal{V} of feedback strategies and a maximum principle will be given.

Obviously $\mathcal{V} \subset \mathcal{U}$. However, as shown earlier (see [1, 2, 4, 5]), for many optimization problems the class \mathcal{V} is sufficiently rich with respect to \mathcal{U} in the sense that a solution of the optimization problem in \mathcal{V} is also one of the optimization problems in \mathcal{U} . In Section 2, we show that this sufficiency of \mathcal{V} with respect to \mathcal{U} remains true for our problem. The maximum principle in the class \mathcal{V} is given in Section 3 and is a generalization of the schedule in [1] with the finite set I .

2. OPTIMIZATION PROBLEM WITH FEEDBACK CONTROLS

Let (Ω, \mathcal{F}, P) be a basic probabilistic space.

The dynamics of the system under consideration consists of three sequences of random variables $x_t = x_t(\omega)$, $u_t = u_t(\omega)$, $\eta_t = \eta_t(\omega)$, $t = 0, 1, 2, \dots, N$.

The system states x_t , $0 \leq t \leq N$, are defined by a difference system

$$x_{t+1} = f_t(x_t, u_t, \eta_t), \quad x_0 = x \in R^n. \quad (2.1)$$

The disturbance states η_t , $0 \leq t \leq N - 1$, take on values from a finite set $I = \{1, 2, \dots, S\}$ and we make the hypothesis that $\{\eta_t\}$ is a controlled Markov chain, i.e., that for every function $l: I \mapsto [0, +\infty)$

$$E\{l(\eta_{t+1}) \mid \eta_t, \eta_{t-1}, \dots, \eta_0\} = \sum_{j=1}^S l(j)p(\eta_t, j, u_{t+1}) \quad (2.2)$$

$$El(\eta_0) = \sum_{j=1}^S l(j)p(j, u_0),$$

where $p(i, j, v)$, $v \in U \subset R^m$ is the transition probability from the state i into the state j when the control parameter equals v and $p(i, v)$, $i \in I$ is the initial probability when the control parameter equals v .

The control parameters u_t , $0 \leq t \leq N - 1$, are functions of the observed system states and disturbances states

$$u_t = u_t(x_0, x_1, \dots, x_t, \eta_0, \eta_1, \dots, \eta_{t-1}) \in U. \quad (2.3)$$

The sequence of functions $u_t: (R^n)^{t+1} \times I^t \mapsto U$ constitutes a nonanticipating control strategy.

In this paper we consider the control parameters of the form

$$u_t = M_t(x_t, q_t, \eta_{t-1}), \quad (2.4)$$

where $M_t: R^{n+1} \times I \rightarrow U$ and q_t is defined by the recurrence equations

$$\begin{aligned} q_{t+1} &= p(i_{t-1}, i_t, u_t) q_t \\ q_1 &= p(i_0, u_0) q_0, \quad q_0 = 1. \end{aligned} \quad (2.5)$$

Obviously, for a given control $u = (u_t, 0 \leq t \leq N-1)$,

$$q_t = q_t^u(i_0, i_1, \dots, i_{t-1}) = P\{\eta_0 = i_0, \eta_1 = i_1, \dots, \eta_{t-1} = i_{t-1}\}$$

defines the probability measure on the set $I^t = I \times I \times \dots \times I$ (t times).

The sequence of functions $M_t(x, q, i)$, $1 \leq i \leq S$, $0 \leq t \leq N-1$ is called a Markov feedback strategy.

A control process $\{x_t, \eta_t, u_t\}$ is said to be admissible if the constraints

$$\varphi_0(x_0, u_0) \leq 0, \varphi_t(x_t, u_t, \eta_{t-1}) \leq 0, 1 \leq t \leq N-1, (q_t^u - \text{a.s.}) \quad (2.6)$$

are satisfied, where $\varphi_t: R^n \times U \times I \mapsto R$, $\varphi_0: R^n \times U \mapsto R$ are given and the measure q_t^u corresponding to the control strategy $u = (u_t, 0 \leq t \leq N-1)$ is given by (2.5).

Denote by $\{x, u \in \mathcal{U}, q^u\} = \{x_t, u_t \in \mathcal{U}, q_t^u\}$ with u_t given by (2.3) an admissible control process and by $\{x, u \in \mathcal{V}, q^u\}$ with u_t given by (2.4) an admissible Markov control process.

The optimization problem with Markov strategies is finding among all admissible Markov control processes $\{x, u \in \mathcal{V}, q^u\}$ an admissible Markov control process $\{x^*, u^* \in \mathcal{V}, q^* = q^{u^*}\}$ such that the performance function

$$J(u, x_0) = E^u \sum_{t=0}^{N-1} g_t(x_t, u_t, \eta_{t-1}) \quad (2.7)$$

attains its minimum value at $\{x^*, u^* \in \mathcal{V}, q^*\}$. Here $\eta_{-1} = 0$ and by E^u we denote the expectation with respect to the probability q^u .

We show now a relation between admissible control processes $\{x, u \in \mathcal{U}, q^u\}$ and $\{x, u \in \mathcal{V}, q^u\}$.

First we need the following lemma.

LEMMA 1. *Let there be a sum $\sum_{i_0^{t-1} \in I^t} \phi(i_0^{t-1}, u(i_0^{t-1}))$, where $i_0^{t-1} = (i_0, i_1, \dots, i_{t-1}) \in I^t = I \times I \times \dots \times I$ (t times),*

$$u: I^t \mapsto U \subset R^m, \quad \phi: I^t \times U \rightarrow R.$$

Then for any mapping $Y: I^t \mapsto R^{n+1}$ and any set-valued mapping $\Gamma: Y(I^t) \mapsto 2^U$ such that

$$u(i_0^{t-1}) \in \Gamma(Y(i_0^{t-1})) \subset U, \quad i_0^{t-1} \in I^t,$$

there exists a function $\varphi = \varphi(y): Y(I^t) \mapsto U$ such that

$$\begin{aligned} \varphi(Y(i_0^{t-1})) &\in \Gamma(Y(i_0^{t-1})) \\ \sum_{i_0^{t-1} \in I^t} \phi(i_0^{t-1}, \varphi(Y(i_0^{t-1}))) &\leq \sum_{i_0^{t-1} \in I^t} \phi(i_0^{t-1}, u(i_0^{t-1})). \end{aligned}$$

Proof. Let \bar{y} belong to $Y(I^t)$, which means there exists $i_0^{t-1} \in I^t$ such that $\bar{y} = Y(i_0^{t-1})$.

Put

$$\begin{aligned} Y^{-1}(\bar{y}) &= \{i_0^{t-1} \mid Y(i_0^{t-1}) = \bar{y}\}, \\ U(\bar{y}) &= \{u(i_0^{t-1}) \in \Gamma(Y(i_0^{t-1})) \mid i_0^{t-1} \in Y^{-1}(\bar{y})\}, \\ V(\bar{y}) &= \left\{ \bar{u} \in U(\bar{y}) \mid \sum_{i_0^{t-1} \in Y^{-1}(\bar{y})} \phi(i_0^{t-1}, \bar{u}) \right. \\ &\quad \left. = \min_{u \in U(\bar{y})} \sum_{i_0^{t-1} \in Y^{-1}(\bar{y})} \phi(i_0^{t-1}, u) \right\} \\ \varphi(\bar{y}) &= \begin{cases} \bar{u} \in V(\bar{y}) & \text{if } \bar{y} \in Y(I^t) \\ u \in U & \text{if } \bar{y} \in R^{n+1} \setminus Y(I^t). \end{cases} \end{aligned}$$

On the other hand

$$I^t = \bigcup_{\bar{y} \in Y(I^t)} Y^{-1}(\bar{y}).$$

This implies that

$$\begin{aligned} \sum_{i_0^{t-1} \in I^t} \phi(i_0, i_1, \dots, i_{t-1}, u(i_0, i_1, \dots, i_{t-1})) \\ = \sum_{\bar{y} \in Y(I^t)} \sum_{i_0^{t-1} \in Y^{-1}(\bar{y})} \phi(i_0^{t-1}, u(i_0^{t-1})) \\ \geq \sum_{\bar{y} \in Y(I^t)} \sum_{i_0^{t-1} \in Y^{-1}(\bar{y})} \phi(i_0^{t-1}, \bar{u} = \varphi(\bar{y})) \end{aligned}$$

$$= \sum_{i_0^{t-1} \in I^t} \phi(i_0^{t-1}, \varphi(Y(i_0^{t-1}))).$$

The proof is complete.

From the lemma the following theorem on the sufficiency of the subclass \mathcal{V} of Markov strategies (2.4) for the class \mathcal{U} of nonanticipating strategies (2.3) follows.

THEOREM 1. *For every admissible control process*

$$(x, u \in \mathcal{U}, q^u) = (\{x_t\}, \{u_t = u_t(i_0, i_1, \dots, i_{t-1})\}, \{q_t^u\}),$$

there exists an admissible process $(\bar{x}, \bar{u} \in \mathcal{V}, q^{\bar{u}}) = (\{\bar{x}_t\}, \{\bar{u}_t\}, \{q_t^{\bar{u}}\})$ with Markov feedback $\bar{u}_t = M_t(\bar{x}_t, \bar{q}_t, \eta_{t-1})$ satisfying the condition

$$E^{\bar{u}} \sum_{t=0}^{N-1} g_t(\bar{x}_t, \bar{u}_t, \eta_{t-1}) \leq E^u \sum_{t=0}^{N-1} g_t(x_t, u_t, \eta_{t-1}).$$

Proof. Theorem 1 is proved by backward induction as in [1]. Assume that Theorem 1 is proved at the k th step; this means we had the functions $\bar{v}_k(x, q, i), \bar{v}_{k+1}(x, q, i), \dots, \bar{v}_{N-1}(x, q, i)$ such that for the admissible control process $\{\bar{x}_t, \bar{u}_t = \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}), q_t^{\bar{u}}, t \geq k+1\}$ on the horizon $\{k, k+1, \dots, N\}$, the relations

- (a) $\bar{x}_{t+1} = f_t(\bar{x}_t, \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}), \eta_t)$
 $\bar{q}_{t+1} = p(\eta_{t-1}, \eta_t, \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}))\bar{q}_t, \quad k \leq t \leq N-1$
 $\bar{x}_k = x_k, \quad \bar{q}_k = q_k^u$
- (b) $\varphi_t(\bar{x}_t, \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}), \eta_{t-1}) \leq 0, \quad k \leq t \leq N-1, (q^{\bar{u}} - \text{a.s.})$
- (c) $E^{\bar{u}} \sum_{t=k}^{N-1} g_t(\bar{x}_t, \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}), \eta_{t-1}) \leq E^u \sum_{t=k}^{N-1} g_t(x_t, u_t, \eta_{t-1})$

are satisfied or relation (a) and the relations (equivalent to (b) and (c))

- (b') $\bar{q}_t \varphi_t(\bar{x}_t, \bar{v}_t(\bar{x}_t, \bar{q}_t, i_{t-1}), i_{t-1}) \leq 0 \quad \text{for all } i_0^{t-1} \in I^t, k \leq t \leq N-1$

$$\begin{aligned}
& (c') \\
& \sum_{i_0^{k-1} \in I^k} q_1^u(i_0) q_2^u(i_0^1) \cdots q_{k-1}^u(i_0^{k-2}) q_k^u(i_0^{k-1}) \\
& \cdot \left\{ \sum_{t=k}^{N-1} \sum_{i_k^{t-1} \in I^{t-k}} \bar{q}_t(i_0^{k-1}, i_k^{t-1}) \right. \\
& \quad \cdot g_t(\bar{x}_t(i_0^{k-1}, i_k^{t-1}); \bar{v}_t(\bar{x}_t(i_0^{k-1}, i_k^{t-1}), \bar{q}_t(i_0^{k-1}, i_k^{t-1}), i_{t-1}); i_{t-1}) \left. \right\} \\
& \leq \sum_{i_0^{k-1} \in I^k} q_1^u(i_0) q_2^u(i_0^1) \cdots q_{k-1}^u(i_0^{k-2}) q_k^u(i_0^{k-1}) \\
& \cdot \left\{ \sum_{t=k}^{N-1} \sum_{i_k^{t-1} \in I^{t-k}} q_t^u(i_0^{k-1}, i_k^{t-1}) \right. \\
& \quad \cdot g_t(x_t(i_0^{k-1}, i_k^{t-1}), u_t(i_0^{k-1}, i_k^{t-1}), i_{t-1}) \left. \right\}
\end{aligned}$$

are satisfied, where (i_0^{k-1}, i_k^{t-1}) stands for i_0^{t-1} and $q^u = (q_1^u, \dots, q_k^u = \bar{q}_k, \bar{q}_{k+1}, \dots, \bar{q}_{N-1})$.

To complete the proof of Theorem 1 it is now enough to show functions $\bar{v}_t(x, q, i)$, $k-1 \leq t \leq N-1$, such that for the admissible control process $\{\bar{x}_t, \bar{u}_t = \bar{v}_t(\bar{x}_t, \bar{q}_t, \eta_{t-1}), q_t^u, t \geq k\}$ on the horizon $\{k-1, k, k+1, \dots, N\}$, relations (a), (b), and (c) or (a), (b'), and (c') are satisfied.

For instance, we have to prove that (see (c'))

$$\begin{aligned}
& (d) \\
& \sum_{i_0^{k-2} \in I^{k-1}} q_1^u(i_0) q_2^u(i_0^1) \cdots q_{k-1}^u(i_0^{k-2}) \\
& \cdot \left\{ \sum_{t=k-1}^{N-1} \sum_{i_{k-1}^{t-1} \in I^{t-k+1}} \bar{q}_t(i_0^{k-2}, i_{k-1}^{t-1}) \right. \\
& \quad \cdot g_t(\bar{x}_t(i_0^{k-2}, i_{k-1}^{t-1}), \bar{v}_t(\bar{x}_t(i_0^{k-2}, i_{k-1}^{t-1}), \bar{q}_t(i_0^{k-2}, i_{k-1}^{t-1}), i_{t-1}); i_{t-1}) \left. \right\} \\
& \leq \sum_{i_0^{k-2} \in I^{k-1}} q_1^u(i_0) q_2^u(i_0^1) \cdots q_{k-1}^u(i_0^{k-2})
\end{aligned}$$

$$\cdot \left\{ \sum_{t=k-1}^{N-1} \sum_{i_{k-1}^{t-1} \in I^{t-k+1}} q^u(i_0^{k-2}, i_{k-1}^{t-1}) \cdot g_t(x_t(i_0^{k-2}, i_{k-1}^{t-1}), u_t(i_0^{k-2}, i_{k-1}^{t-1}), i_{t-1}) \right\}.$$

Introduce the notations

$$F(X_{k-1}(i_0^{k-2}), w, i_{k-2}) = \sum_{i_{k-1}, \dots, i_{N-1}=1}^S \left[g_{k-1}(X_{k-1}(i_0^{k-2}), w, i_{k-2}) + \sum_{t=k}^{N-1} Q_t(w) g_t(X_t(w), v_t(X_t(w), Q_t(w), i_{t-1}), i_{t-1}) \right],$$

where the sequence of random variables $\{X_k(w), Q_k(w), X_t(w) = X_t(k, X_k(w), Q_k(w)), Q_t(w) = Q_t(k, X_k(w), Q_k(w)), k+1 \leq t \leq N\}$ depends on the parameter $w \in U$ and is defined by the relations

$$\begin{aligned} X_{t+1}(w) &= f_t(X_t(w), \bar{v}_t(X_t(w), Q_t(w), \eta_{t-1}), \eta_t) \\ Q_{t+1}(w) &= p(\eta_{t-1}, \eta_t, \bar{v}_t(X_t(w), Q_t(w), \eta_{t-1})) Q_t(w), \quad k \leq t \leq N-1 \\ X_k(w) &= f_{k-1}(X_{k-1}, w, \eta_{k-1}), X_{k-1} = x_{k-1} \\ Q_k(w) &= p(\eta_{k-2}, \eta_{k-1}, w) Q_{k-1}, Q_{k-1} = q_{k-1}. \end{aligned}$$

Note that these relations are well defined because the functions $\bar{v}_t(x, q, i)$, $t \geq k$, are defined by the induction.

Consider the sets

$$\begin{aligned} \Gamma_t(x_{k-1}, q_{k-1}^u, i_{k-1}) &= \{w \in U \mid \bar{q}_t \varphi_t(X_t(w), \bar{v}_t(X_t(w), Q_t(w), i_{t-1}), i_{t-1}) \\ &\leq 0, i_{k-1}^{t-1} \in I^{t-k+1}\}, \quad k \leq t \leq N-1, \\ \Gamma_{k-1}(x_{k-1}, q_{k-1}^u, i_{k-2}) &= \{w \in U \mid q_{k-1}^u \varphi_{k-1}(x_{k-1}, w, i_{k-2}) \leq 0\} \\ \Gamma &= \bigcap_{t=k-1}^{N-1} \Gamma_t. \end{aligned}$$

The set Γ is nonempty since $u_{k-1} \in \Gamma$. Moreover, these sets Γ_t , $k-1 \leq t \leq N-1$, and Γ depend only on x_{k-1} , q_{k-1}^u , i_{k-2} .

Let us apply Lemma 1 to the following functions and mappings:

$$\begin{aligned} \phi(i_0^{k-2}, u_{k-1}(i_0^{k-2})) &= q_1^u(i_0) q_2^u(i_0^1) \cdots q_{k-2}^u(i_0^{k-3}) q_{k-1}^u(i_0^{k-2}) \\ &\cdot F(x_{k-1}(i_0^{k-2}), q_{k-1}^u(i_0^{k-2}), u_{k-1}(i_0^{k-2}), i_{k-2}), \end{aligned}$$

$$Y = (x_{k-1}(i_0^{k-2}), q_{k-1}^u(i_0^{k-2}), i_{k-2}): I^{k-1} \mapsto R^n \times R \times I \subset R^{n+2}$$

$$\Gamma: Y(I^{k-1}) \mapsto 2^U, \quad u_{k-1}(i_0^{k-2}): I^{k-1} \mapsto \Gamma(Y(I^{k-1})) (\subset U).$$

Then we obtain a function $\bar{v}_{k-1}(x, q, i)$ such that

$$\bar{v}_{k-1}(x, q, i) \in U,$$

$$\bar{v}_{k-1}(x_{k-1}(i_0^{k-2}), q_{k-1}^u(i_0^{k-2}), i_{k-2}) \in \Gamma(Y(I^{k-1})) \subset U,$$

$$\sum_{i_0, \dots, i_{k-2}=1}^S \phi(i_0^{k-2}, \bar{v}_{k-1}(x_{k-1}(i_0^{k-2}), q_{k-1}(i_0^{k-2}), i_{k-2}))$$

$$\leq \sum_{i_0, \dots, i_{k-2}=1}^S \phi(i_0^{k-2}, u_{k-1}(i_0^{k-2})).$$

The proof of (d) is complete. By putting

$$\bar{x}_k = X_k(\bar{v}_{k-1}(x_{k-1}, q_{k-1}, i_{k-1})), \quad \bar{q}_k = Q_k(\bar{v}_{k-1}(x_{k-1}, q_{k-1}, i_{k-1})),$$

$$\bar{x}_t = X_t(\bar{v}_{k-1}(x_{k-1}, q_{k-1}, i_{k-1})), \quad \bar{q}_t = Q_t(\bar{v}_{k-1}(x_{k-1}, q_{k-1}, i_{k-1})),$$

the relations (a), (b'), and (c'), replacing k by $k-1$, remain true. The initial verification for $k = N-1$ follows from the fact that we can construct $M_{N-1}(x, q, i)$ such that $\bar{u}_{N-1} = u_{N-1}$. Theorem 1 is proved.

3. MAXIMUM PRINCIPLE FOR MARKOV STRATEGIES

We consider the optimization problem (2.1), (2.2), (2.4), (2.6), (2.7) with the following assumptions on the functions f_t , φ_t , g_t and the set $U \subset R_m$ (see [7]).

(A) Functions g_t, f_t, φ_t are continuous in (x, u) , differentiable with respect to x and their derivatives are continuous in x .

(B) For every $x \in R^n$, $u' \in U$, $\alpha \in [0, 1]$, $i \in I$, $u'' \in U$, there exists an element $u = u(x, u', u'', \alpha, i) \in U$ such that the following relations hold:

$$g_t(x, u, i) \leq \alpha g_t(x, u', i) + (1 - \alpha) g_t(x, u'', i),$$

$$f_t(x, u, j) = \alpha f_t(x, u', j) + (1 - \alpha) f_t(x, u'', j), \quad 1 \leq j \leq S,$$

$$p(i, j, u) = \alpha p(i, j, u') + (1 - \alpha) p(i, j, u''), \quad 1 \leq j \leq S,$$

$$\varphi_t(x, u, i) \leq \alpha \varphi_t(x, u', i) + (1 - \alpha) \varphi_t(x, u'', i).$$

(C) There exist functions $\bar{x}_t: I^t \mapsto R^n$, $\bar{q}_t: I^t \mapsto R$, $\bar{u}_t: I^t \mapsto U$ such that for every $i_0^{-1} = (i_0, i_1, \dots, i_{t-1}) \in I^t$,

$$\bar{x}_{t+1} = \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i_t) \bar{x}_t + f_t(x_t^*, \bar{u}_t, i_t) - f_t(x_t^*, u_t^*, i_t), \bar{x}_0 = 0, \quad (3.1)$$

$$\bar{q}_{t+1} = p(i_{t-1}, i_t, u_t^*) \bar{q}_t + p(i_{t-1}, i_t, \bar{u}_t) q_t^* - p(i_{t-1}, i_t, u_t^*) q_t^*, \quad (3.2)$$

$$\bar{q}_0 = 0, \quad 0 \leq t \leq N-1,$$

and for every $r = 1, 2, \dots, k$, $i_0^{t-1} \in I^t$ with $q_t^* \varphi_t^r(x_t^*, u_t^*, i_{t-1}) = 0$,

$$\begin{aligned} q_t^* \frac{\partial}{\partial x} \varphi_t^r(x_t^*, u_t^*, i_{t-1}) \bar{x}_t + q_t^* \varphi_t^r(x_t^*, \bar{u}_t, i_{t-1}) \\ + \varphi_t^r(x_t^*, u_t^*, i_{t-1}) \bar{q}_t < 0 \quad (q^* - \text{a.s.}). \end{aligned}$$

Here we denote by $(\{x_t^* = x_t^*(i_0^{t-1})\}, \{u_t^* = u_t^*(i_0^{t-1})\}, q^* = q^{u^*})$ the optimal control process.

To formulate the main result we introduce the Hamiltonian functions

$$\begin{aligned} H_{t+1}(i_{t-1}, i_t, \psi, \chi, \lambda, (x, q), u) = \frac{1}{S} q \cdot g_t(x, u, i_{t-1}) \\ - \psi' \cdot f_t(x, u, i_t) - \chi \cdot p(i_{t-1}, i_t, u) q + \frac{1}{S} q \cdot \lambda' \cdot \varphi_t(x, u, i_{t-1}), \quad (3.3) \end{aligned}$$

where $i_{t-1} \in I$, $i_t \in I$, $q: I^t \mapsto [0, 1]$, $\psi: I^t \mapsto R^n$, $\chi: I^t \mapsto R$, $\lambda: I^t \mapsto (R^+)^k$, $R^+ = [0, +\infty)$, S is the number of states of Markov disturbances, and prime denotes the transpose

THEOREM 2. *If Assumptions (A)–(C) hold and*

$$\{x^*, u^*, q^{u^*}\} = \{\{x_t^*\}, \{u_t^* = M_t^*(x_t^*, q_t^*, \eta_{t-1})\}, \{q_t^*\}\}$$

is an optimal admissible Markov control process, then there exist functions $\psi_t = \psi_t(x_{t-1}^, q_{t-1}^*, \eta_{t-2}, \eta_{t-1})$, $\chi_t = \chi_t(x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1})$, $\lambda_t = \lambda_t(x_t^*, q_t^*, \eta_{t-1})$ with values in R^n , R , and $(R^+)^k$, respectively, such that the function of variable u (see (3.3))*

$$\begin{aligned} \mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u) = \sum_{i_t=1}^S H_{t+1}(i_{t-1}, i_t, \psi_{t+1} \\ \times (x_t^*, q_t^*, i_{t-1}, i_t), \chi_{t+1}(x_t^*, q_t^*, i_{t-1}, i_t), \lambda_t(x_t^*, q_t^*, i_{t-1}), x_t^*, q_t^*, u) \end{aligned}$$

*attains its minimum values at the point u_t^**

$$\min_{u \in U} \mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u) = \mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u_t^* = M_t^*(x_t^*, q_t^*, i_{t-1})).$$

Moreover the functions ψ_t, x_t, λ_t satisfy the following relations:

$$\begin{aligned}
 E^{u^*}[\psi_t | x_t^*, q_t^*, \eta_{t-1}] &= -q_t^* \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, \eta_{t-1}) \\
 &\quad + \sum_{i=1}^S \psi_{t+1}(x_t^*, q_t^*, \eta_{t-1}, i) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i) \\
 &\quad - \lambda_t q_t^* \frac{\partial}{\partial x} \varphi_t(x_t^*, u_t^*, \eta_{t-1}), \\
 q_t^* E^{u^*}[\chi_t | x_t^*, q_t^*, \eta_{t-1}] &= -q_t^* g_t(x_t^*, u_t^*, \eta_{t-1}) \\
 &\quad + \sum_{i=1}^S \chi_{t+1}(x_t^*, q_t^*, \eta_{t-1}, i) p(\eta_{t-1}, i, u_t^*) q_t^*, \\
 \psi_N &= 0, \quad \chi_N = 0, \\
 \lambda_t q_t^*(x_t^*, q_t^*, i_{t-1}) &= 0 \quad \text{for all } i_0^{t-1} \in I^t.
 \end{aligned}$$

Proof. From Theorem 1, it follows that the Markov control process (x^*, u^*, q^*) is also optimal in the class of all nonanticipating control strategies (2.3).

Hence we can apply the following maximum principle (see [7], Theorem 1) to derive necessary conditions for the optimal control process (x^*, u^*, q^*) in this class of all strategies (2.3).

LEMMA 2. *If Assumptions (A)–(C) hold and (x^*, u^*, q^*) is an optimal admissible control process for the problem (2.1)–(2.3), (2.5)–(2.7) with nonanticipating strategies (2.3), then there exist $\psi_t = \psi_t(i_0^{t-1})$, $\chi_t = \chi_t(i_0^{t-1})$, $\lambda'_t = \lambda'_t(i_0^{t-1}) = (\lambda'_t(i_0^{t-1}), 1 \leq r \leq k)$ such that the following relations are satisfied for all $i_0^{t-1} \in I^t$:*

$$\lambda'_t(i_0^{t-1}) \geq 0, \quad 1 \leq r \leq k, \quad (3.4)$$

$$\begin{aligned}
 \psi_t &= -q_t^* \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, i_{t-1}) + \sum_{i_t=1}^S \psi_{t+1}(i_0^t) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i_t) \\
 &\quad - \lambda_t q_t^* \frac{\partial}{\partial x} \varphi_t(x_t^*, u_t^*, i_{t-1})
 \end{aligned} \quad (3.5)$$

$$\begin{aligned}
 q_t^* \chi_t &= -q_t^* g_t(x_t^*, u_t^*, i_{t-1}) \\
 &\quad + \sum_{i_t=1}^S \chi_{t+1}(i_0^{t-1}, i_t) \cdot p(i_{t-1}, i_t, u_t^*) q_t^*
 \end{aligned} \quad (3.6)$$

$$q_t^* \cdot \lambda_t' \cdot \varphi_t(x_t^*, u_t^*, i_{t-1}) = 0 \quad (3.7)$$

$$\mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u_t^* = M_t^*(x_t^*, q_t^*, i_{t-1})) \leq \mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u_t(i_0^{t-1})) \quad (3.8)$$

for all $u_t(i_0^{t-1}) \in U$.

Note that in the proof of Lemma 2 (see [7, Theorem 1]) we put

$$\psi_t(i_0^{t-1}) = 0, \quad \chi_t(i_0^{t-1}) = 0 \quad \text{for all } i_0^{t-1} \in I^t \setminus A_t^*, \quad (3.9)$$

where $A_t^* = \{i_0^{t-1} \in I^t : q_t^{u^*}(i_0^{t-1}) > 0\}$, $1 \leq t \leq N$.

If we consider only the Markov control strategies (2.4) with control parameters of the form $u_t = M_t(x_t^*, q_t^*, \eta_{t-1})$ then (3.8) gives us

$$\begin{aligned} \mathcal{H}_t(x_t^*, q_t^*, \eta_{t-1}, u_t^* = M_t^*(x_t^*, q_t^*, \eta_{t-1})) \\ \leq \mathcal{H}_t(x_t^*, q_t^*, \eta_{t-1}, u_t(\eta_0^{t-1}) = M_t(x_t^*, q_t^*, \eta_{t-1})), \end{aligned} \quad (3.10)$$

where $(\eta_0, \eta_1, \dots, \eta_{t-1})$ is a Markov chain with the transition probabilities $q_0, q_1^* = q_1^{u^*}, \dots, q_{t-1}^* = q_{t-1}^{u^*}$.

For brevity we denote (see (3.3))

$$H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \lambda_t, u) = H_{t+1}(i_{t-1}, i_t, \psi_{t+1}, \chi_{t+1}, \lambda_t, x_t^*, q_t^*, u). \quad (3.11)$$

From the definition of \mathcal{H}_t and (3.9) we have

$$\mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u) = \sum_{i \in I^*} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \lambda_t, u),$$

where

$$\begin{aligned} I^* &= \{i \in I \mid q_{t+1}^{u^*}(i_0^{t-1}, i) > 0\} \\ &= \{i \in I \mid q_t^u(i_0^{t-1}) > 0, p(i_{t-1}, i, u_t^*(i_0^{t-1})) > 0\}. \end{aligned} \quad (3.12)$$

Then

$$\mathcal{H}_t(x_t^*, q_t^*, \eta_{t-1}, u) = E \left\{ \frac{1}{\gamma} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \lambda_t, u) \mid \eta_0, \eta_1, \dots, \eta_{t-1} \right\}$$

and from (3.10) it follows that

$$\begin{aligned} E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \lambda_t, u_t^*) \mid x_t^*, q_t^*, \eta_{t-1} \right] \\ \leq E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \lambda_t, M_t(x_t^*, q_t^*, \eta_{t-1})) \mid x_t^*, q_t^*, \eta_{t-1} \right], \end{aligned} \quad (3.13)$$

where $\gamma = 1/p(\eta_{t-1}, \eta_t, u_t^*)$ and by $E^{u^*}[\cdot]$ we denote the conditional expectation with respect to the probability measure q^{u^*} .

Since H_{t+1}^* is linear in λ_t , (3.13) remains true in replacing λ_t by $\bar{\lambda}_t = E^{u^*}(\lambda_t | x_t^*, q_t^*, \eta_{t-1})$ and then

$$\begin{aligned} & E^{u^*} \left\{ E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \bar{\lambda}_t, u_t^*) | x_t^*, q_t^*, \eta_{t-1}, \eta_t \right] | x_t^*, q_t^*, \eta_{t-1} \right\} \\ & \leq E^{u^*} \left\{ E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\psi_{t+1}, \chi_{t+1}, \bar{\lambda}_t, \right. \right. \\ & \quad \left. \left. M_t(x_t^*, q_t^*, \eta_{t-1}) \right) | x_t^*, q_t^*, \eta_{t-1}, \eta_t \right] | x_t^*, q_t^*, \eta_{t-1} \right\}. \end{aligned}$$

Replacing ψ_{t+1}, χ_{t+1} by

$$\bar{\psi}_{t+1} = \bar{\psi}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, \eta_t) = E^{u^*}[\psi_{t+1} | x_t^*, q_t^*, \eta_{t-1}, \eta_t] \quad (3.14)$$

$$\bar{\chi}_{t+1} = \bar{\chi}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, \eta_t) = E^{u^*}[\chi_{t+1} | x_t^*, q_t^*, \eta_{t-1}, \eta_t] \quad (3.15)$$

we rewrite the last inequality

$$\begin{aligned} & E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\bar{\psi}_{t+1}, \bar{\chi}_{t+1}, \bar{\lambda}_t, u_t^* = M_t^*(x_t^*, q_t^*, \eta_{t-1})) | x_t^*, q_t^*, \eta_{t-1} \right] \\ & \leq E^{u^*} \left[\frac{1}{\gamma} H_{t+1}^*(\bar{\psi}_{t+1}, \bar{\chi}_{t+1}, \bar{\lambda}_t, u_t = M_t(x_t^*, q_t^*, \eta_{t-1})) | x_t^*, q_t^*, \eta_{t-1} \right]. \end{aligned}$$

Also, from replacing $\psi_t(i_0^{t-1}), \chi_t(i_0^{t-1}), \lambda_t(i_0^{t-1})$ by

$$\bar{\psi}_t(x_t^*, q_t^*, \eta_{t-1}, \eta_t), \bar{\chi}_t(x_t^*, q_t^*, \eta_{t-1}, \eta_t), \bar{\lambda}_t(x_t^*, q_t^*, \eta_{t-1})$$

it follows that the Hamiltonian functions H_{t+1} of the variable u (see (3.3)) depend only on i_{t-1}, i_t, x_t, q_t .

By the relation $\sigma(x_t^*, q_t^*, \eta_{t-1}) \subset \sigma(\eta_0, \eta_1, \dots, \eta_{t-1})$ we obtain the following form of the last inequality:

$$\begin{aligned} & \sum_{i \in I^*} H_{t+1}(i_{t-1}, i, \bar{\psi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \bar{\chi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \\ & \quad \bar{\lambda}_t(x_t^*, q_t^*, i_{t-1}), x_t^*, q_t^*, u_t^*) \\ & \leq \sum_{i \in I^*} H_{t+1}(i_{t-1}, i, \bar{\psi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \bar{\chi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \\ & \quad \bar{\lambda}_t(x_t^*, q_t^*, i_{t-1}), x_t^*, q_t^*, M_t(x_t^*, q_t^*, i_{t-1})) \\ & \quad \text{for all } i_0^{t-1} \in A_t^*. \end{aligned}$$

For all $(i_0^{t-1}, i) \notin A_{t+1}^*$, which means either $q_t^* = 0$ or $q_t^* \neq 0$ but $p(i_{t-1}, i_t, M_t^*(x_t^*, q_t^*, i_{t-1})) = 0$, we put

$$\bar{\psi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i) = 0, \quad \bar{\chi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i) = 0$$

and we put $\bar{\lambda}_t(x_t^*, q_t^*, i_{t-1}) = 0$ if $q_t^* = 0$.

Hence the last inequality becomes

$$\begin{aligned} & \sum_{i=1}^S \left[H_{t+1}(i_{t-1}, i, \bar{\psi}_{t+1} \right. \\ & \quad \times (x_t^*, q_t^*, i_{t-1}, i), \bar{\chi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \bar{\lambda}_t(x_t^*, q_t^*, i_{t-1}), x_t^*, q_t^*, u_t^*) \Big] \\ & \leq \sum_{i=1}^S \left[H_{t+1}(i_{t-1}, i, \bar{\psi}_{t+1}(x_t^*, q_t^*, i_{t-1}, i), \bar{\chi}_{t+1} \right. \\ & \quad \times (x_t^*, q_t^*, i_{t-1}, i), \bar{\lambda}_t(x_t^*, q_t^*, i_{t-1}), x_t^*, q_t^*, M_t(x_t^*, q_t^*, i_{t-1})) \Big] \end{aligned}$$

for all $M_t(x_t^*, q_t^*, i_{t-1}) \in U$. Thus $\bar{\psi}_{t+1}, \bar{\chi}_{t+1}, \bar{\lambda}_t$ satisfy the first assertion of Theorem 2.

The last assertion of Theorem 2 for $\bar{\lambda}_t$ follows from (3.7). To show adjoint systems for $\bar{\psi}_t, \bar{\chi}_t$ we use (3.5), (3.6).

By (3.9), the relations (3.5), (3.6) can be rewritten

$$\begin{aligned} \psi_t &= -q_t^* \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, \eta_{t-1}) \\ &+ E^{u^*} \left[\frac{1}{\gamma} \psi_{t+1}(i_0^t) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, \eta_t) \mid x_t^*, q_t^*, \eta_{t-1} \right] \\ &- \lambda_t q_t^* \frac{\partial}{\partial x} \varphi_t(x_t^*, u_t^*, \eta_{t-1}) \\ q_t^* \chi_t &= -q_t^* g_t(x_t^*, u_t^*, \eta_{t-1}) \\ &+ E^{u^*} \left[\frac{1}{\gamma} \chi_{t+1} p_t(\eta_{t-1}, \eta_t, u_t^*) q_t^* \mid x_t^*, q_t^*, \eta_{t-1} \right], \end{aligned}$$

where $\gamma = 1/p(\eta_{t-1}, \eta_t, u_t^*)$.

Since $\sigma(x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1}) \supset \sigma(x_t^*, q_t^*, \eta_{t-1})$ we have

$$\begin{aligned} & E^{u^*} \{ E^{u^*} [\psi_t \mid x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1}] \mid x_t^*, q_t^*, \eta_{t-1} \} \\ &= -q_t^* \cdot \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, \eta_{t-1}) \end{aligned}$$

$$\begin{aligned}
& + E^{u^*} \left\{ E^{u^*} \left[\frac{1}{\gamma} \psi_{t+1} \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, \eta_t) \mid x_t^*, q_t^*, \eta_{t-1}, \eta_t \right] \mid x_t^*, q_t^*, \eta_{t-1} \right\} \\
& - E^{u^*} [\lambda_t \mid x_t^*, q_t^*, \eta_{t-1}] q_t^* \frac{\partial}{\partial x} \varphi_t(x_t^*, u_t^*, \eta_{t-1}), \\
& E^{u^*} \{ E^{u^*} [\chi_t \mid x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1}] \mid x_t^*, q_t^*, \eta_{t-1} \} \\
& = -q_t^* g_t(x_t^*, u_t^*, \eta_{t-1}) \\
& + E^{u^*} \left\{ E^{u^*} \left[\frac{1}{\gamma} \chi_{t+1} p(\eta_{t-1}, \eta_t, u_t^*) \right. \right. \\
& \quad \left. \left. \times q_t^* \mid x_t^*, q_t^*, \eta_{t-1}, \eta_t \right] \mid x_t^*, q_t^*, \eta_{t-1} \right\}.
\end{aligned}$$

Hence (see (3.14), (3.15))

$$\begin{aligned}
& E^{u^*} [\bar{\psi}_t \mid x_t^*, q_t^*, \eta_{t-1}] \\
& = -q_t^* \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, \eta_{t-1}) \\
& + E^{u^*} \left\{ \frac{1}{\gamma} \bar{\psi}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, \eta_t) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, \eta_t) \mid x_t^*, q_t^*, \eta_{t-1} \right\} \\
& - \bar{\lambda}_t q_t^* \frac{\partial}{\partial x} \varphi_t(x_t^*, u_t^*, \eta_{t-1}), \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
q_t^* E^{u^*} \{ \bar{\lambda}_t \mid x_t^*, q_t^*, \eta_{t-1} \} & = -q_t^* g_t(x_t^*, u_t^*, \eta_{t-1}) \\
& + E^{u^*} \left\{ \frac{1}{\gamma} \bar{\lambda}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, \eta_t) \right. \\
& \quad \left. \cdot p(\eta_{t-1}, \eta_t, u_t^*) q_t^* \mid x_t^*, q_t^*, \eta_{t-1} \right\}.
\end{aligned}$$

From the fact that

$$\begin{aligned}
& E^{u^*} \left\{ \frac{1}{\gamma} \bar{\psi}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, \eta_t) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, \eta_t) \mid x_t^*, q_t^*, \eta_{t-1} \right\} \\
& = \sum_{i=1}^S \bar{\psi}_{t+1}(x_t^*, q_t^*, \eta_{t-1}, i) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i)
\end{aligned}$$

and (3.16) it follows that $\bar{\psi}_t$, $t = N - 1, N - 2, \dots$, satisfy the adjoint system for ψ_t in Theorem 2.

The adjoint system for χ_t is deduced in the same way and Theorem 2 is proved.

Remark. The adjoint system for χ_t is also of the form

$$\begin{aligned} q_t^* \chi_t &= -q_t^* g_t(x_t^*, u_t^*, \eta_{t-1}) \\ &+ \sum_{i=1}^S \chi_{t+1}(x_t^*, q_t^*, \eta_{t-1}, i) p(\eta_{t-1}, i, u_t^*) q_t^*. \end{aligned}$$

Indeed, by induction, from (3.6) it follows that χ_t is $\sigma(x_t^*, q_t^*, \eta_{t-1})$ -measurable. Hence $\chi_t = \chi_t(x_t^*, q_t^*, \eta_{t-1})$ and

$$\begin{aligned} \bar{\chi}_t &= E^{u^*}[\chi_t | x_{t-1}^*, q_{t-1}^*, \eta_{t-1}, \eta_{t-2}] \\ &= E^{u^*}[\chi_t(f_{t-1}(x_{t-1}^*, u_{t-1}^* \\ &\quad \times (x_{t-1}^*, q_{t-1}^*, \eta_{t-1}), \eta_{t-1}), p(\eta_{t-2}, \eta_{t-1}, u_{t-1}^*(x_{t-1}^*, q_{t-1}^*, \eta_{t-1})) \\ &\quad \times q_{t-1}^*, \eta_{t-1}) | x_{t-1}^*, q_{t-1}^*, \eta_{t-1}, \eta_{t-2}] \\ &= \chi_t(x_t^*, q_t^*, \eta_{t-1}). \end{aligned}$$

Thus $\bar{\chi}_t \equiv \chi_t$ and from the adjoint system for χ_t in the proof of Theorem 2 the assertion of this remark follows.

For the optimization problem (2.1), (2.2), (2.4), (2.7) without the constraints (2.6) ($\varphi_t = 0$), the adjoint system (3.5) without the last component is analogous to (3.6). In the same way as in the proof of Remark, we obtain the following adjoint system for ψ_t :

$$\begin{aligned} \psi_t &= -q_t^* \frac{\partial}{\partial x} g_t(x_t^*, u_t^*, \eta_{t-1}) \\ &+ \sum_{i=1}^S \psi_{t+1}(x_t^*, q_t^*, \eta_{t-1}, i) \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i). \end{aligned} \quad (3.17)$$

THEOREM 3. Let $\{x_t^*, u_t^* = M_t^*(x_t^*, q_t^{u^*}, \eta_{t-1}), q_t^* = q_t^{u^*}\}$ be an optimal admissible Markov control process for the problem (2.1), (2.2), (2.4), (2.7) without the constraints (2.6).

Then there exist $\psi_t = \psi_t(x_t^*, q_t^*, \eta_{t-1}) = \psi_t(x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1})$, $\chi_t = \chi_t(x_t^*, q_t^*, \eta_{t-1}) = \chi_t(x_{t-1}^*, q_{t-1}^*, \eta_{t-2}, \eta_{t-1})$, $1 \leq t \leq N - 1$, such that the adjoint system for χ_t in Remark and the adjoint system for ψ_t (3.17) are satisfied and the function of variable $u \mathcal{H}_t(x_t^*, q_t^*, i_{t-1}, u)$ attains its minimum value at the point u_t^* .

4. CONCLUSION

Design in a changing environment involves the optimal control problem with two features: control-dependent Markov disturbances and feedback strategies. To support decisions we have shown a maximum principle under these conditions and affirmed that the adjoint processes can be made measurable on $(x_t, q_t, \eta_{t-1}, \eta_t)$ rather than on the whole past. The efficient formula for the adjoint processes is provided in the case of the problem without constraints.

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